

# Bode Plots Revisited: a Software System for Automated Generation of Piecewise Linear Frequency Response Plots

Predrag Pejović and Amela Zeković

**Abstract**—The paper presents an algorithm for automatic generation of piecewise linear Bode plots. The algorithm is complete in the sense it covers for all possible locations of poles and zeros of transfer functions, including unstable poles and poles and zeros at the imaginary axis. The starting transfer function is factored into a canonical form, and thirteen elementary transfer function types are defined by their canonical forms. The thirteen elementary transfer function types are shown to be derived from just five generic transfer function types, and piecewise linear Bode plots are defined and depicted for all five of the generic types. For all thirteen elementary transfer function types the nodes they introduce in the piecewise linear plots are specified, as well as the algorithms how they affect the node altitudes. Finally, a three stage algorithm that produces both the Bode plots and the exact numerically computed frequency response plots is described. The algorithm is implemented in a command line based program, illustrated in a filter example, and future work directions are indicated, aiming a graphical user interface and integration of the program to a linear system symbolic analysis software suite.

*Original Research Paper*  
DOI: 10.7251/ELSI721076P

## I. INTRODUCTION

Electric circuit design was the first discipline where computer aided analysis became widely spread. Pioneering work of [1], expanded and updated in [2], is still a foundation stone upon which the analysis tools were built. In its basis, the approach of [1], [2] is numerical, which results in reduced efforts to build circuit and the cost of playing with their parameters. Frequency response plots, which are focused in this paper, are covered in [1], [2] within the AC analysis, and frequency response of linear or linearized circuits is readily obtained as the program output. In the control systems area, which applies Bode plots [3] in the frequency response method to design control loops [4], [5], computer aided design tools provide amplitude and phase plots through various packages, like the control package for GNU Octave [6]. Both of the approaches result in smooth numerically computed curves, corresponding to exact frequency response equations. The results are accurate, but they lack clear information needed for the design: which parameters to tune to obtain desired response?

The design oriented analysis is pioneered by R. D. Middlebrook [7], [8], where presentation of mathematically obtained data in a convenient and intuitive manner is focused in order to facilitate effective and creative system synthesis. The approach

is expanded and documented in [9], [10], and [11]. Aim of this paper is to follow this path, and to formalize and completely cover the Bode plots in an algorithmic manner. Under the term “Bode plots” so called “asymptotic” plots are assumed. Although the authors agree that the terminological issues are less relevant, we must state that “asymptotic plot” is the term accurate only for some of the “elementary transfer functions”, that are going to be defined formally somewhat later. In some of the cases, the linear plot is exact, and there are no asymptotes there. On the other hand, in some other cases asymptotes constitute only a part of the corresponding Bode plot for the elementary transfer function. However, in all of the cases, the resulting Bode plot is piecewise linear. Thus, under “Bode plot” we will assume piecewise linear approximation of the amplitude and phase response of a considered transfer function or an impedance. The term “transfer function” will be used both for actual transfer functions of linear systems or for the network impedances [9], [10] and admittances.

Classical circuit theory textbooks [12], [13] address Bode plots, but primarily in examples and in a rather intuitive fashion, focusing to cases frequently encountered in practice. Some other textbook examples, not to be cited here, provide either incomplete, either incomplete and overly simplified coverage of the topic. Some of the approaches are so rough to prevent estimating the phase margin, which is essential in the control loop design. Aim of this paper is to provide complete algorithmic approach to creating piecewise linear approximations of the amplitude and phase frequency response, understood under the term “Bode plots”. The approach is algorithmic enough and complete only when it can be programmed and when all possible situations are covered, including unstable transfer functions, which do exist in control systems and need to be stabilized, and resonant cases with poles on the imaginary axis, resulting in infinite values in the logarithmic amplitude response plot. To program the proposed algorithm, Python programming language [14] will be used, accompanied by PyLab modules NumPy [15], SciPy [16], and matplotlib [17]. The choice is made due to high level programming capabilities of Python and its modules, with advanced data types and powerful list processing methods, additionally requiring all of the used software tools to be free software.

To summarize, aim of the paper is to provide an algorithm to generate piecewise linear frequency response Bode plots for amplitude and phase of linear systems with lumped parameters. The algorithm is required to be complete, to cover all possible locations of poles and zeros. Motivation to design such algorithm is in the design oriented analysis, to provide simplified and clear information how specific discrete values

of design variables affect the frequency response, in order to facilitate efficient frequency response shaping during the design process.

## II. PROBLEM STATEMENT

Let us consider a transfer function  $H(s)$  of a linear lumped parameter system. Its frequency response is a complex function  $H(j\omega)$  for  $\omega \geq 0$ , representing values of  $H(s)$  at the upper ray of the imaginary axis. As the system being considered is a lumped parameter one,  $H(s)$  is a rational function of complex frequency  $s$ , and it can be factored and written in a canonical form

$$H(s) = K \prod_{i=1}^{k_z} (H_{z,i}(s))^{l_{z,i}} \prod_{i=1}^{k_p} (H_{p,i}(s))^{l_{p,i}} \quad (1)$$

where  $K \in \mathbb{R}$  and  $k_z, k_p, l_{z,i}, l_{p,i} \in \mathbb{N}$ . The factorization into a canonical form is algorithmic, though the choice of the canonical form is somewhat arbitrary, and the choice criterion applied in this paper is that the canonical form should be intuitive. All aspects of the canonical form are going to be formally specified somewhat later, and at this point it is sufficient to note that the factorization is performed into a constant  $K$ , zero producing elementary transfer functions  $H_{z,i}(s)$ , and pole producing elementary transfer functions  $H_{p,i}(s)$ . Each of the zero producing and pole producing elementary functions produces either one zero or pole, either a pair of complex conjugate zeros or poles.

Aim of the algorithm is to generate piecewise linear approximations of the amplitude response

$$a(\omega) = 20 \log |H(j\omega)| \quad (2)$$

and the phase response

$$\varphi(\omega) = \arg(H(j\omega)) \quad (3)$$

For phase response plots of the elementary transfer functions we generally assume  $\varphi \in [-180^\circ, 180^\circ]$ , with some exceptions introduced by mirroring of elementary transfer functions containing pole or zero pairs. For the entire transfer function phase response plot, the curve has not been reduced to this range for the sake of its readability, though the reduction could be easily performed in a straightforward manner.

Substituting (1) into (2) in order to get the logarithmic amplitude response, the products of elementary transfer functions transform to sums of their logarithmic equivalents

$$a(\omega) = 20 \log |K| + \sum_{i=1}^{k_z} l_{z,i} 20 \log |H_{z,i}(s)| + \sum_{i=1}^{k_p} l_{p,i} 20 \log |H_{p,i}(s)| \quad (4)$$

and substituting (1) into (3) transforms the products to sums of phase responses of elementary transfer functions

$$\varphi(\omega) = \arg(K) + \sum_{i=1}^{k_z} l_{z,i} \arg(H_{z,i}(s)) + \sum_{i=1}^{k_p} l_{p,i} \arg(H_{p,i}(s)). \quad (5)$$

Thus, the task of creating amplitude and phase plot of the frequency response is reduced to summing frequency responses of elementary transfer functions.

The choice of elementary transfer functions is somewhat arbitrary. Some textbooks do not provide complete coverage of the complex plane by allowed places of poles and zeros, focusing to frequently encountered responses only, like restricting the attention to stable poles, or even to real axis only. In this paper, to provide complete coverage and an intuitive set of elementary transfer functions for the canonical form, the following list of elementary transfer functions is proposed:

- 1) constant,  $H_a(s)$
- 2) pole at the origin,  $H_b(s)$
- 3) zero at the origin,  $H_c(s)$
- 4) stable real pole,  $H_d(s)$
- 5) left half plane real zero,  $H_e(s)$
- 6) unstable real pole,  $H_f(s)$
- 7) right half plane real zero,  $H_g(s)$
- 8) stable pair of poles,  $H_h(s)$
- 9) left half plane pair of zeros,  $H_i(s)$
- 10) unstable pair of poles,  $H_j(s)$
- 11) right half plane pair of zeros,  $H_k(s)$
- 12) pair of poles at the imaginary axis,  $H_l(s)$
- 13) pair of zeros at the imaginary axis,  $H_m(s)$ .

The set of elementary transfer functions is not minimal. Allowing negative exponents  $l_{z,i}$  and  $l_{p,i}$  would reduce the set from thirteen elementary transfer functions to seven. Furthermore, allowing  $\pm$  sign in some of the elementary transfer functions would further reduce the set to only five transfer functions, which is the minimal set if we exclude infinite  $Q$ -factors as an option (otherwise the set would be reduced to only four transfer functions, but numerical computation problems with infinite  $Q$ -factor values would emerge). The choice is made following the logic of the design oriented analysis, to provide an intuitive set, and to provide information how each elementary transfer function affects the frequency response.

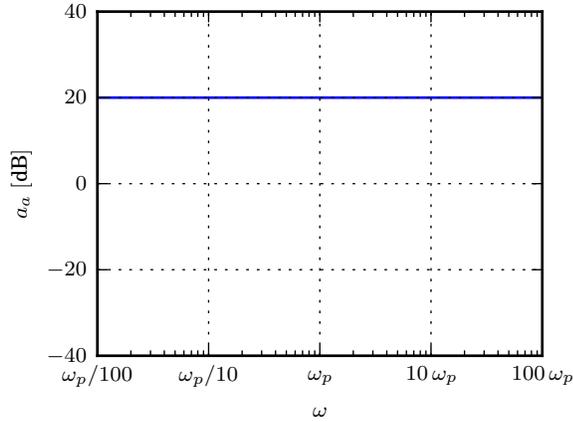
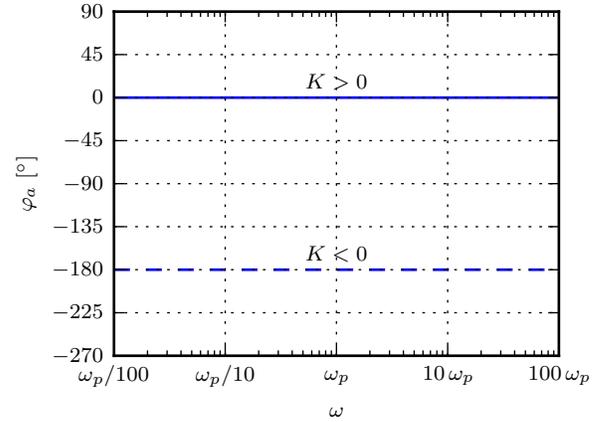
## III. FREQUENCY RESPONSES OF ELEMENTARY TRANSFER FUNCTIONS

In this section, thirteen elementary transfer functions will be defined by specifying their canonical forms and their frequency responses in terms of piecewise linear approximation, which sometimes really is asymptotic, but sometimes not only asymptotic. For the five “really elementary” transfer functions, the plots will be provided, while for the others appropriate sign changing relations, i.e. mirroring, will be indicated, being sufficient to provide complete description of the plots. Critical points of the elementary transfer functions, named nodes, in which line segments of the the piecewise linear curves change their directions, will be defined, as well as the effects they cause to other nodes, both in the amplitude and in the phase response.

### A. Constant

The first elementary transfer function is a constant,

$$H_a(a) = K \quad (6)$$

Fig. 1.  $H_a(s)$ , amplitude response.Fig. 2.  $H_a(s)$ , phase response.

where  $K \in \mathbb{R}$ . For linear lumped parameter systems the constant is necessarily real, and it could be either positive or negative. For the constant equal to zero there is not much to plot, thus this case is excluded from the analysis. Also, in the case  $K = 1$  it would be assumed that this elementary transfer function is not present in the product of elementary transfer functions (1), since it would not produce any change in the frequency response plots.

Amplitude response of the constant is

$$a_a(\omega) = 20 \log |K| \quad (7)$$

and it only shifts all of the amplitude response nodes in the vertical direction (modifies so called “altitudes”), not introducing any nodes of its own. The response is depicted in Fig. 1.

The phase response of the constant depends on its sign, and it is

$$\varphi_a(\omega) = \begin{cases} 0, & \text{for } K > 0 \\ -\pi, & \text{for } K < 0. \end{cases} \quad (8)$$

Again, new nodes are not introduced, and the phase response is either not affected, for  $K > 0$ , either is shifted down for  $180^\circ$  for  $K < 0$  for all of the phase response nodes. The phase response for both of the cases is depicted in Fig. 2.

It is worth to mention that in this case the plot is exact, there are no approximations being introduced, there are no asymptotes.

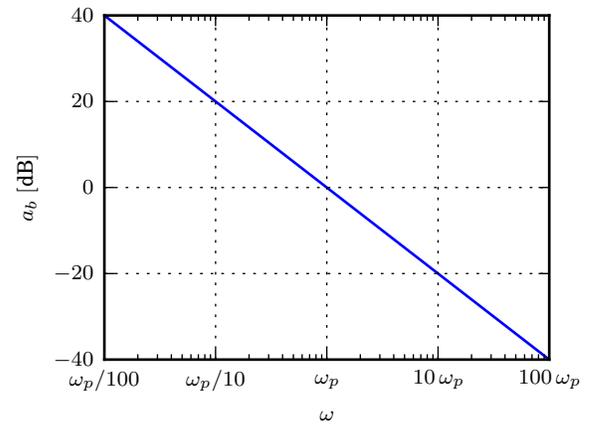
### B. Pole at the origin

A frequent case encountered in practice is to have a pole at the origin. The elementary transfer function is specified by its canonical form

$$H_b(s) = \frac{\omega_p}{s} \quad (9)$$

where  $\omega_p > 0$ , resulting in the amplitude response

$$a_b(\omega) = -20 \log \frac{\omega}{\omega_p} \quad (10)$$

Fig. 3.  $H_b(s)$ , amplitude response.

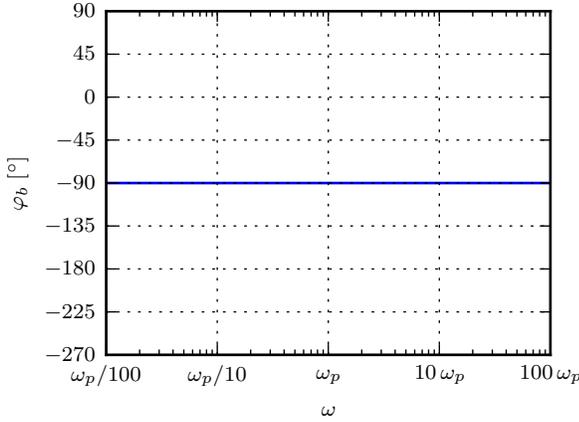
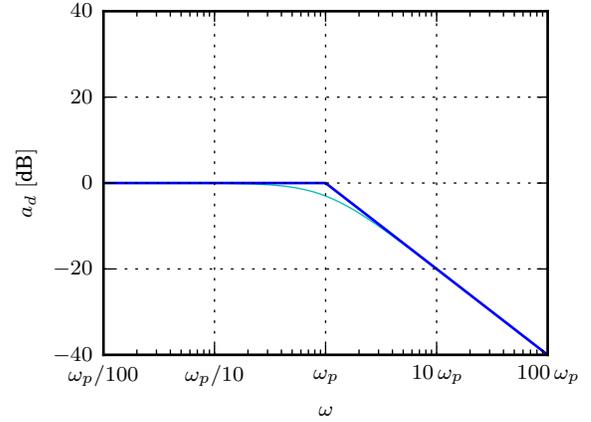
and the phase response

$$\varphi_b(\omega) = -\frac{\pi}{2}. \quad (11)$$

As for the constant, in this case both the amplitude and the phase response are exact, there are no approximations being introduced. Besides, they are linear, there are no changes in the piecewise linear curve direction, and new nodes do not have to be introduced. The amplitude response of this elementary transfer function is depicted in Fig. 3, while the phase response is depicted in Fig. 4.

For the amplitude response, this elementary transfer function increases the level of the amplitude response at each node for  $20 \text{ dB} \log \frac{\omega_p}{\omega}$ , where  $\omega$  is the angular frequency of the node. The overall phase response is affected by this elementary transfer function such that for all nodes the phase is reduced by  $90^\circ$ .

In this case, the piecewise linear plot, which happened to be linear, is exact, approximations are not required.

Fig. 4.  $H_b(s)$ , phase response.Fig. 5.  $H_d(s)$ , amplitude response.

### C. Zero at the origin

The next elementary transfer function is zero at the origin, specified by its canonical form

$$H_c(s) = \frac{s}{\omega_z} \quad (12)$$

where  $\omega_z > 0$ , which results in the amplitude response

$$a_c(\omega) = 20 \log \frac{\omega}{\omega_z} \quad (13)$$

and the phase response

$$\varphi_c(\omega) = \frac{\pi}{2}. \quad (14)$$

Again, the piecewise linear plot is exact and purely linear. This elementary transfer function does not introduce any nodes. To implement automatic adjustment of node amplitudes and phases, it is worth to note that  $a_c(\omega) = -a_b(\omega)$  and  $\varphi_c(\omega) = -\varphi_b(\omega)$  assuming parameter  $\omega_z$  of  $H_c(s)$  having value equal to the value of the parameter  $\omega_p$  of  $H_b(s)$ .

### D. Stable real pole

The next elementary transfer function to be considered is the stable real pole, i.e. the pole in the left half plane. This elementary transfer function is characterized by

$$H_d(s) = \frac{1}{1 + \frac{s}{\omega_p}} \quad (15)$$

where  $\omega_p > 0$ , which results in the amplitude response of

$$a_d(\omega) = -10 \log \left( 1 + \left( \frac{\omega}{\omega_p} \right)^2 \right) \quad (16)$$

and the phase response of

$$\varphi_d(\omega) = -\arctan \left( \frac{\omega}{\omega_p} \right). \quad (17)$$

In this case, the piecewise linear representation is approximate. For the amplitude response, it is entirely asymptotic, as depicted in Fig. 5. For the phase response, the piecewise linear

representation is asymptotic in two out of three segments, as depicted in Fig. 6, while the third segment is somewhat arbitrary, to connect the parallel asymptotes, as detailed in [9], connecting the asymptotes by a line segment from  $\omega_p/10$  to  $10\omega_p$ .

Regarding the nodes, the amplitude response adds a node at  $\omega_p$ . Before that frequency, for  $\omega \leq \omega_p$ , this elementary transfer function does not affect the overall amplitude response. After that frequency, for  $\omega_p < \omega$ , the nodes are shifted down for  $20 \text{ dB} \log \frac{\omega}{\omega_p}$ .

The phase response adds nodes at  $\omega_p/10$  and at  $10\omega_p$ . For  $\omega \leq \omega_p/10$ , the overall phase response is not affected. For  $\omega_p/10 \leq \omega < 10\omega_p$ , the overall phase response is affected such that the phase is reduced by  $45^\circ \log \frac{10\omega}{\omega_p}$ . For  $10\omega_p \leq \omega$ , the overall phase response is affected such that the phase is shifted down for  $90^\circ$ . Contributions of this elementary transfer function to overall amplitude and phase response are depicted in Figs. 5 and 6.

The piecewise linear representation of this elementary transfer function is approximate. As detailed in [9], the maximum of the error caused by the approximation is 3 dB for the amplitude response, at  $\omega_p$ , while the maximum of the error for the phase response is about  $6^\circ$ . In Figs. 5 and 6 the exact responses are plotted in thin lines.

approximate

### E. Left half plane real zero

Opposite type of the elementary transfer function to the stable real pole is the left half plane real zero, caused by a zero of the transfer function (1) at  $s = -\omega_z$ . For  $\omega_p$  of  $H_d(s)$  being equal to  $\omega_z$  of  $H_e(s)$ , the two elementary transfer functions would cancel out, thus they are the opposites. The elementary transfer function  $H_e(s)$  in its canonical form is

$$H_e(s) = 1 + \frac{s}{\omega_z} \quad (18)$$

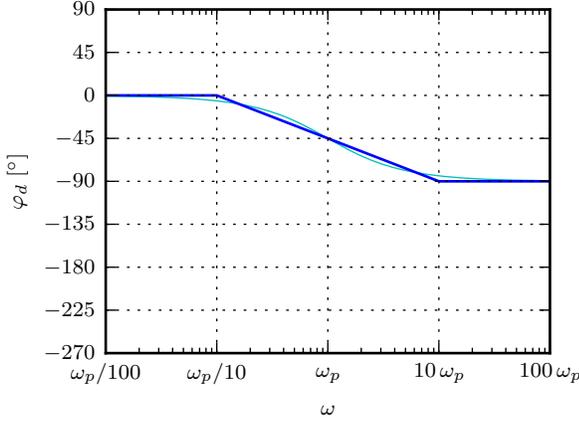


Fig. 6.  $H_d(s)$ , phase response.

where  $\omega_z > 0$ , resulting in the amplitude response of

$$a_e(\omega) = 10 \log \left( 1 + \left( \frac{\omega}{\omega_z} \right)^2 \right) \quad (19)$$

and the phase response of

$$\varphi_e(\omega) = \arctan \left( \frac{\omega}{\omega_z} \right). \quad (20)$$

Relations of this elementary transfer function responses to the responses of  $H_d(s)$  are given by  $a_e(\omega) = -a_d(\omega)$  and  $\varphi_e(\omega) = -\varphi_d(\omega)$ , for the parameter values  $\omega_z = \omega_p$ . Thus, these responses need not to be plotted here, they are just mirrored responses shown in Figs. 5 and 6. This elementary transfer function adds nodes at  $\omega_z$  for the amplitude response and  $\omega_z/10$  and  $10\omega_z$  for the phase response. The responses are approximate in the same manner as the responses of  $H_d(s)$ .

#### F. Unstable real pole

This elementary transfer function is frequently omitted from consideration in popular textbooks, since stable open loop systems are focused. However, such response is possible in real systems, an example of such is the peak limiting current mode controlled buck converter in the discontinuous conduction mode for the steady state duty ratio of  $D > \frac{1}{2}$ . The canonical form of the elementary transfer function is

$$H_f(s) = \frac{1}{1 - \frac{s}{\omega_p}} \quad (21)$$

where  $\omega_p > 0$ , resulting in the amplitude response of

$$a_f(\omega) = -10 \log \left( 1 + \left( \frac{\omega}{\omega_p} \right)^2 \right) \quad (22)$$

and the phase response of

$$\varphi_f(\omega) = \arctan \left( \frac{\omega}{\omega_p} \right). \quad (23)$$

Relation of  $H_f(s)$  to  $H_d(s)$  is  $a_f(\omega) = a_d(\omega)$  and  $\varphi_f(\omega) = -\varphi_d(\omega)$ , meaning that the amplitude responses are equal,

while the phase responses are mirrored. The elementary transfer function  $H_f(s)$  introduces the same critical points as  $H_d(s)$ , and introduces the same level of approximation.

#### G. Right half plane real zero

A complement of unstable real pole is the left half plane real zero elementary transfer function, specified by the transfer function canonical form

$$H_g(s) = 1 - \frac{s}{\omega_z} \quad (24)$$

where  $\omega_z > 0$ , which results in the amplitude response of

$$a_g(\omega) = 10 \log \left( 1 + \left( \frac{\omega}{\omega_z} \right)^2 \right) \quad (25)$$

and the phase response of

$$\varphi_g(\omega) = -\arctan \left( \frac{\omega}{\omega_z} \right). \quad (26)$$

For parameter values  $\omega_z = \omega_p$ , relation of the responses of  $H_g(s)$  to the responses of  $H_d(s)$  are given by  $a_g(\omega) = -a_d(\omega)$ ,  $\varphi_g(\omega) = \varphi_d(\omega)$ . The elementary transfer function adds the same nodes as  $H_d(s)$ , and introduces the same level of approximation.

Elementary transfer functions  $H_d(s)$ ,  $H_e(s)$ ,  $H_f(s)$ , and  $H_g(s)$  form a group of real axis excluding origin poles and zeros, sharing the same nodes and similar amplitude and phase responses. They could have been treated as a single response type, but in this approach they are treated as four distinct elementary transfer functions to underline different effects they cause to the frequency response.

#### H. Stable pair of poles

The next group containing four elementary transfer functions starts with the pair of complex conjugate poles, specified by a canonical form of the elementary transfer function

$$H_h(s) = \frac{1}{1 + \frac{s}{Q_p \omega_p} + \frac{s^2}{\omega_p^2}} \quad (27)$$

where  $\omega_p > 0$  and  $Q_p > 0$ . The assumption that the elementary transfer function represents a complex conjugate pair of poles results in  $Q_p > \frac{1}{2}$ . This elementary transfer function results in the amplitude response

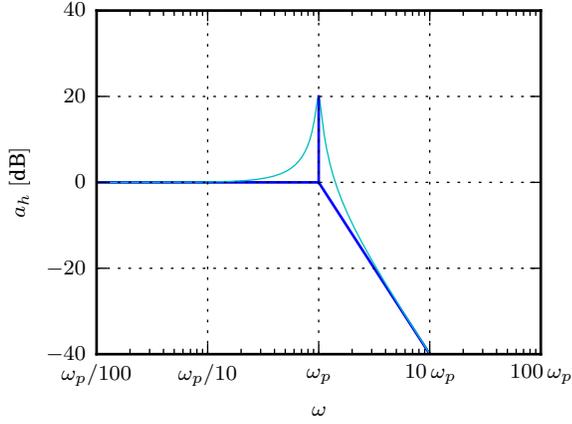
$$a_h(\omega) = -10 \log \left( \left( 1 - \left( \frac{\omega}{\omega_p} \right)^2 \right)^2 + \left( \frac{\omega}{Q_p \omega_p} \right)^2 \right) \quad (28)$$

and the phase response

$$\varphi_h(\omega) = -\arctan \left( \frac{1}{Q_p} \frac{\omega \omega_p}{\omega_p^2 - \omega^2} \right). \quad (29)$$

The responses are dependent on two parameters,  $\omega_p$  and  $Q_p$ , resulting in somewhat more complex piecewise linear plotting rules. The rules are detailed in [9], and the resulting plots are depicted in Figs. 7 and 9.

For the amplitude response, the elementary transfer function introduces a node at  $\omega_p$ . For  $\omega \leq \omega_p$ , the overall amplitude

Fig. 7.  $H_h(s)$ ,  $Q_p = 10$ , amplitude response.

response is not affected. For  $\omega_p < \omega$ , the amplitude response is reduced by  $40 \text{ dB} \log \frac{\omega}{\omega_p}$ . Besides that, the amplitude response includes an overshoot which at  $\omega = \omega_p$  equals  $q_p = 20 \log Q_p$ . In the piecewise linear representation, this overshoot is represented by a vertical line segment at  $\omega_p$  of  $q_p$  in height, as depicted in Fig. 7.

For the phase response, let us introduce the transition zone parameter according to [9] first, as

$$r = 10^{\frac{1}{2Q_p}}. \quad (30)$$

This parameter is dependent on the  $Q$ -factor  $Q_p$  and specifies the nodes and thus the phase response transition zone. The dependence of  $r$  on  $Q_p$  is depicted in Fig. 8, and it sharply decreases from  $r = 10$  for  $Q_p = \frac{1}{2}$  to its asymptotic value of 1 as  $Q_p$  increases. The nodes are introduced at  $\omega_p/r$  and at  $r\omega_p$ . For  $\omega \leq \omega_p/r$  the overall phase response is not affected. For  $\omega_p/r < \omega \leq r\omega_p$  the phase response is reduced by  $\frac{90^\circ}{\log r} \log \frac{r\omega}{\omega_p}$ . For  $r\omega_p < \omega$  the overall phase response is reduced by  $180^\circ$ . The phase response of this elementary transfer function is shown in Fig. 9.

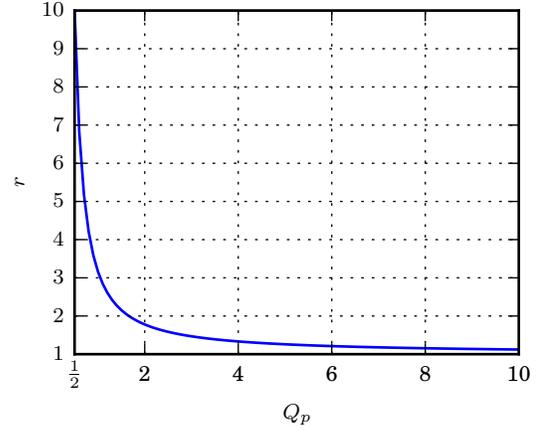
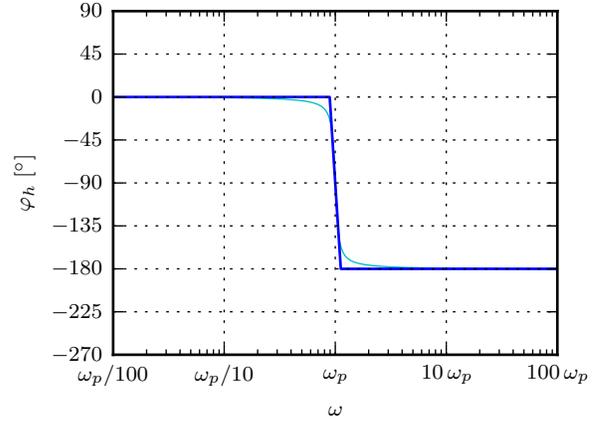
For  $Q_p = \frac{1}{2}$ , the complex conjugate pair reduces to a repeated pole at the real axis, and both the amplitude and the phase response of the piecewise linear Bode plot are equal for both of the approaches, providing consistency of the approximate analysis.

As already indicated, both the amplitude response and the phase response are approximate, and the error is dependent on  $Q_p$ . In Figs. 7 and 9 the exact responses are plotted in thin lines.

#### I. Left half plane pair of zeros

As already indicated, the elementary transfer function  $H_h(s)$  is a basis for a group of four elementary transfer functions, like  $H_d(s)$  is the basis for the group that also involves  $H_e(s)$ ,  $H_f(s)$ , and  $H_g(s)$ . The first of the elementary transfer functions based upon  $H_h(s)$  is its opposite

$$H_i(s) = 1 + \frac{s}{Q_z \omega_z} + \frac{s^2}{\omega_z^2} \quad (31)$$

Fig. 8. Dependence of  $r$  on  $Q_p$ .Fig. 9.  $H_h(s)$ ,  $Q_p = 10$ , phase response.

where  $\omega_z > 0$  and  $Q_z > 0$ , resulting in the amplitude response

$$a_i(\omega) = 10 \log \left( \left( 1 - \left( \frac{\omega}{\omega_z} \right)^2 \right)^2 + \left( \frac{\omega}{Q_z \omega_z} \right)^2 \right) \quad (32)$$

and the phase response

$$\varphi_i(\omega) = \arctan \left( \frac{1}{Q_z} \frac{\omega \omega_z}{\omega_z^2 - \omega^2} \right). \quad (33)$$

Assuming the parameter values  $\omega_z = \omega_p$  and  $Q_z = Q_p$ , relation of the responses of  $H_i(s)$  to the responses of  $H_h(s)$  are  $a_i(\omega) = -a_h(\omega)$ , and  $\varphi_i(\omega) = -\varphi_h(\omega)$ . The nodes are the same as for  $H_h(s)$ , and both the amplitude and the phase responses are mirrored.

#### J. Unstable pair of poles

The next in this group of elementary transfer functions is caused by an unstable pair of complex conjugate poles, specified by its canonical form

$$H_j(s) = \frac{1}{1 - \frac{s}{Q_p \omega_p} + \frac{s^2}{\omega_p^2}} \quad (34)$$

where  $\omega_p > 0$  and  $Q_p > 0$ . This elementary transfer function results in the amplitude response

$$a_j(\omega) = -10 \log \left( \left( 1 - \left( \frac{\omega}{\omega_p} \right)^2 \right)^2 + \left( \frac{\omega}{Q_p \omega_p} \right)^2 \right) \quad (35)$$

and the phase response

$$\varphi_j(\omega) = \arctan \left( \frac{1}{Q_p} \frac{\omega \omega_p}{\omega_p^2 - \omega^2} \right). \quad (36)$$

The responses are related to the responses of  $H_h(s)$  by  $a_j(\omega) = a_h(\omega)$  and  $\varphi_j(\omega) = -\varphi_h(\omega)$ , sharing the same nodes, the same amplitude responses, and mirrored phase responses.

### K. Right half plane pair of zeros

The last in this group of elementary transfer functions is specified by its canonical form

$$H_k(s) = 1 - \frac{s}{Q_z \omega_z} + \frac{s^2}{\omega_z^2} \quad (37)$$

where  $\omega_z > 0$  and  $Q_z > 0$ , and results in the amplitude response of

$$a_k(\omega) = 10 \log \left( \left( 1 - \left( \frac{\omega}{\omega_z} \right)^2 \right)^2 + \left( \frac{\omega}{Q_z \omega_z} \right)^2 \right) \quad (38)$$

and the phase response of

$$\varphi_k(\omega) = -\arctan \left( \frac{1}{Q_z} \frac{\omega \omega_z}{\omega_z^2 - \omega^2} \right). \quad (39)$$

Assuming parameter values  $\omega_z = \omega_p$  and  $Q_z = Q_p$ , the responses are related to the responses of  $H_h(s)$  such that  $a_k(\omega) = -a_h(\omega)$ ,  $\varphi_k(\omega) = \varphi_h(\omega)$ . The nodes are the same as for  $H_h(s)$ , the amplitude response is mirrored, while the phase responses are the same.

### L. Pair of poles at the imaginary axis

The last group of elementary transfer functions considers two elementary transfer functions: pair of poles at the imaginary axis and pair of zeros at the imaginary axis. This group of elementary transfer functions had to be separated to avoid numerical problems that would be caused by the infinite value of corresponding  $Q$ -factors.

Pair of poles at the imaginary axis is specified by the elementary transfer function in its canonical form

$$H_l(s) = \frac{1}{1 + \frac{s^2}{\omega_p^2}} \quad (40)$$

where  $\omega_p > 0$ , resulting in the amplitude response of

$$a_l(\omega) = -10 \log \left( 1 - \left( \frac{\omega}{\omega_p} \right)^2 \right)^2 \quad (41)$$

and the phase response of

$$\varphi_l(\omega) = \begin{cases} 0, & \omega < \omega_p \\ -\pi, & \omega_p < \omega. \end{cases} \quad (42)$$

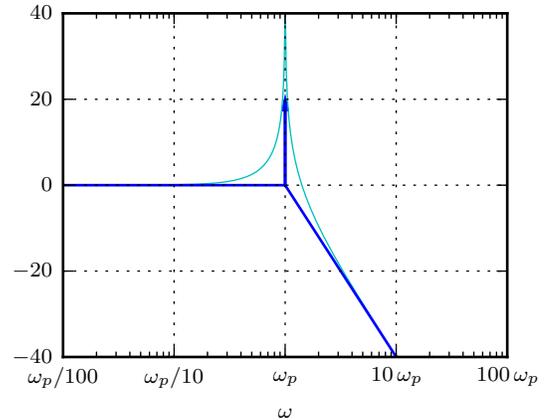


Fig. 10.  $H_l(s)$ , amplitude response.

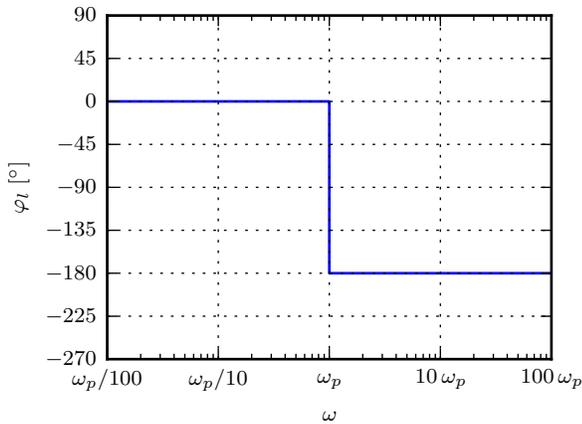
The amplitude response is specific in the sense it contains a singularity for a finite value of  $\omega$ , for  $\omega = \omega_p$ , which is the node for the amplitude response. The piecewise linear representation of the amplitude response is depicted in Fig. 10, and it contains two asymptotes, for  $\omega \rightarrow 0$  and for  $\omega \rightarrow \infty$ . The asymptotes intersect at  $\omega_p$ , which is the node. The piecewise linear representation of this elementary transfer function does not affect the overall amplitude response for  $\omega \leq \omega_p$ , while for  $\omega_p < \omega$  it decreases the amplitude nodes for  $40 \log \frac{\omega}{\omega_p}$ . At  $\omega = \omega_p$  the elementary transfer function has a pole, resulting in the amplitude response singularity, which is indicated by an arrow in Fig. 10, since there is no other way to indicate infinity in the response. The exact amplitude response is indicated by the thin line. The piecewise linear representation is approximate, having significant error at some values of  $\omega$ .

Contrary to the piecewise linear representation of the amplitude response, the piecewise linear representation of the phase response is exact, as depicted in Fig. 11. For  $\omega \leq \omega_p$  the overall phase response is not affected by this elementary transfer function. For  $\omega_p < \omega$ , the overall phase response is affected such the phases for all nodes are decreased for  $180^\circ$ . This response is somewhat peculiar in the sense it is discontinuous. This requires two nodes for the phase response at the same frequency,  $\omega_p$ . At the first of these nodes, in terms of the node index, not frequency, the phase response is not affected; at the second, it is reduced by  $180^\circ$ .

### M. Pair of zeros at the imaginary axis

The last of the elementary transfer functions considers the pair of zeros at the imaginary axis, represented by the elementary transfer function canonical form

$$H_m(s) = 1 + \frac{s^2}{\omega_z^2} \quad (43)$$

Fig. 11.  $H_l(s)$ , phase response.

where  $\omega_z > 0$ , and it is the opposite of  $H_l(s)$ , having the amplitude response

$$a_m(\omega) = 10 \log \left( 1 - \left( \frac{\omega}{\omega_z} \right)^2 \right)^2 \quad (44)$$

and the phase response

$$\varphi_m(\omega) = \begin{cases} 0, & \omega < \omega_z \\ -\pi, & \omega_z < \omega. \end{cases} \quad (45)$$

Assuming the parameter value  $\omega_z = \omega_p$ , relation of the frequency responses of  $H_m(s)$  to the responses of  $H_l(s)$  are  $a_m(\omega) = -a_l(\omega)$  and  $\varphi_m(\omega) = \varphi_l(\omega)$ . The responses of the two elementary transfer functions share the same nodes. The amplitude response is mirrored, while the phase response is the same. Actually, it is mirrored, but restriction that the phase response of elementary transfer functions is within  $[-180^\circ, 180^\circ]$  reduced the phase response to be the same, by shifting down for  $360^\circ$  at  $\omega > \omega_p$ .

#### N. A constraint

After the elementary transfer functions have been defined, in order to provide unique factorization let us introduce a constraint here: in the factorization of (1) only one of the elementary transfer functions  $H_a(s)$ ,  $H_b(s)$ , and  $H_c(s)$  is present.

## IV. THE ALGORITHM FOR CREATING THE BODE PLOTS

The algorithm described in this paper starts with the transfer function factored into canonical form of (1). To achieve the factorization, either exact factorization may be applied using computer algebra tools, like Maxima [18] or SymPy [19], either approximate methods described in [9], [10], [11] might be used. This task is planned to be automated in near future. The factored transfer function of (1) is specified in an input file, where a line corresponds to an elementary transfer function. The elementary transfer functions are specified by several fields. The first field specifies the type of elementary transfer function, and it is followed by one or two fields that

specify the elementary transfer function parameters:  $K$  for  $H_a(s)$ ,  $\omega_p$  or  $\omega_z$  for  $H_b(s)$  to  $H_g(s)$ , and for  $H_l(s)$  and  $H_m(s)$ , while for  $H_h(s)$  to  $H_k(s)$   $\omega_p$  or  $\omega_z$  and  $Q_p$  or  $Q_z$  are specified. The next field contains exponent of the elementary transfer function, labeled by  $l_{p,i}$  and by  $l_{z,i}$  in (1). A frequent value of this parameter is 1. The following field contains a label assigned to the elementary transfer function. Finally, the last two fields contain flags indicating whether the final diagram will contain Bode plot of the elementary transfer function besides the plot of (1) and whether the final plot will contain exact plot of the elementary transfer function.

Besides the lines that specify elementary transfer functions, the input file contains two additional lines to specify values of  $\omega_{min}$  and  $\omega_{max}$  that define the frequency range in which the frequency response is plotted. The final line specifies flags whether the Bode plot and/or the exact plot are included in the final diagram. As a matter of fact, the final diagram needs not to be generated at all: the program might be run just to generate and save the data that the user might use later to create his/her own diagrams.

#### A. Parsing

The first part of the program is a parser, which is considered as an auxiliary part, not the part of the algorithm itself. The parser initialized the data structures and goes through the input file, firstly trying to parse the lines that specify elementary transfer functions. This step involves identifying elementary transfer function type and checking the parameter values such that all  $\omega_p$  and  $\omega_z$  values are positive floats, that  $Q_p$  and  $Q_z$  values are floats greater than  $\frac{1}{2}$ , and that exponents are positive integers. If an elementary function specifying line does not pass a parsing requirements, the program quits and the user is informed about the problem. Similar procedure applies for the final three lines that specify  $\omega_{min}$ ,  $\omega_{max}$ , and the diagram plotting flags. Also there are some auxiliary topics: following the conventions set in [1], the first line of the file contains the transfer function name; comment lines might be added and they are indicated by starting the comment line with #, as in Python and some other programming languages.

The data structure provided by the parser contain lists of parameter values, including the labels and the flags, for each of the elementary transfer function types, as well as the display values of  $\omega_{min}$  and  $\omega_{max}$ , and the diagram plotting flags.

#### B. Setting the nodes

The first pass of the algorithm sets the nodes for the amplitude and the phase response and the frequency ranges both for computing the plot and for displaying the plot. The display range is always a subset of the computing range. To achieve this goal, the program initializes lists of amplitude nodes and phase nodes as empty lists, and passing through the data structure adds the nodes as specified by the elementary transfer function descriptions given in Section III.

After the lists of nodes required by the elementary transfer functions is generated, the task is to determine frequency range for computing the plot. For the union of the node frequencies

for the amplitude plot and the phase plot the minimum and the maximum are determined. To avoid exceptions, it is worth to mention that in the case the only elementary transfer function in (1) is either  $H_a(s)$ ,  $H_b(s)$ , or  $H_c(s)$ , the initial set of node frequencies, before adding the border frequencies, is an empty set; this should be handled by the program smoothly, just by skipping the search and by taking the display border frequencies as the computational border frequencies. Possible candidates for the computing range limits  $\omega_{min}$  and  $\omega_{max}$  are determined as one tenth of the minimal node frequency and ten times the maximal node frequency. If the display frequency range is wider than the range required by the nodes, the values of  $\omega_{min}$  and  $\omega_{max}$  are adjusted to cover the entire display range. Finally, the values of  $\omega_{min}$  and  $\omega_{max}$  are added as the nodes both for the amplitude response and for the phase response. When the values of  $\omega_{min}$  and  $\omega_{max}$  are determined, an array object of frequency values to determine the exact amplitude and phase response is created using the logspace function. The number of data points is determined automatically, depending on  $\omega_{min}$  and  $\omega_{max}$ , to provide about 100 data points per decade.

After the node frequency values are set, the lists of node frequencies are converted to array data type of the NumPy module [15], and associated arrays of the amplitude response values and the phase response values are initialized as objects of the matching length containing zeros. The second pass of the algorithm is required to adjust these values, which essentially creates the piecewise linear Bode plot. The third pass sets the vertical line segments and/or the arrows that correspond to infinite resonant responses.

### C. Adjusting the nodes

The second pass of the algorithm adjusts the nodes, both for the amplitude and the phase response. The algorithm is described in Section III for each of the elementary transfer functions. The program passes through the data structure created by the parser for each of the elementary transfer function types and adjusts the node altitudes as required.

Besides adjusting the node altitudes, the second pass of the algorithm possesses information about  $\omega_{min}$  and  $\omega_{max}$ , thus able to create the database of Bode plots for each of the elementary transfer functions contained in (1), the exact frequency responses for all of the elementary transfer functions, and the exact frequency response of (1) according to (2) and (3).

### D. Resolving resonances

The third pass of the algorithm starts at the point when the Bode plots are almost finished, and only “resonant responses” caused by  $Q_p$  and  $Q_z$  values should be indicated, as well as the infinite values in the logarithmic plots of the amplitude responses caused by  $H_l(s)$  and  $H_m(s)$ . It should be noted that this phase of the algorithm involves amplitude response only. The third pass of the algorithm starts by initializing the lists of line segment patches and the list of arrow patches as empty lists.

The first part of “resolving resonances” considers elementary transfer functions  $H_h(s)$  to  $H_k(s)$  that contribute to the line segment patches. Values of the amplitude response at corresponding value of  $\omega_p$  or  $\omega_z$  are read from the piecewise linear Bode plot of the amplitude response, and having these values and the value of corresponding  $q_p$  or  $q_z$ , the vertical line segment patch is created.

The second part of “resolving resonances” considers elementary transfer functions  $H_l(s)$  and  $H_m(s)$  that create arrow patches. Values of the amplitude response at corresponding value of  $\omega_p$  or  $\omega_z$  are read from the piecewise linear Bode plot of the amplitude response, and these values are used as starting points for the arrows. The arrows point upwards for  $H_l(s)$  and downwards for  $H_m(s)$ . The length of the arrow that indicates infinite response is chosen to be standardized to equivalent 20 dB.

This concludes the algorithm, and all the data that specify both the exact plots and the piecewise linear Bode plots are constructed. If the user requires, an informative plot is created, containing curves specified by the flags. This is convenient in the control loop design process, but publication quality graphs usually require access to the created database of plots, to adjust ticks and grids, and to add some labeling manually, according to the user wishes.

## V. AN EXAMPLE

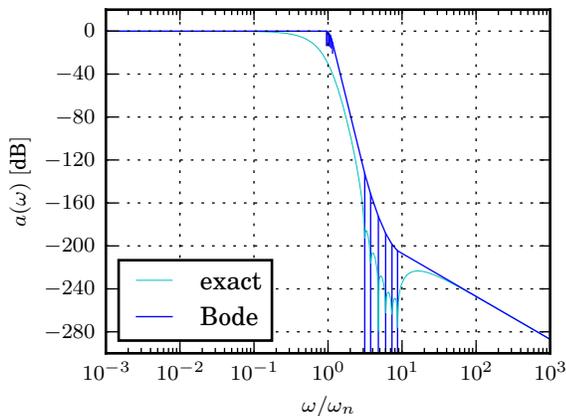
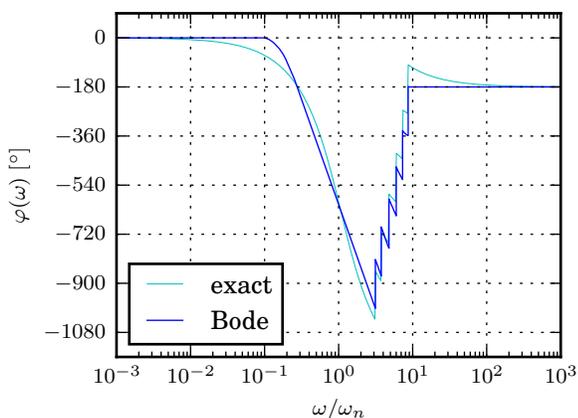
To illustrate application of the described algorithm, an equiripple group delay filter with amplitude corrector, inspired by [20], [21] is selected. The filter contains 14 poles and 12 zeros. Filter applications are not particularly suited for Bode piecewise linear approximation, since both the poles and zeros are closely grouped in the complex plane, resulting in tendency of the approximation error to accumulate. Thus, the example should be considered as a particularly hard test for the method.

Transfer function of the considered filter  $H_{vbl}(s)$  is factored according to (1) into the following list of elementary transfer functions

- 1)  $H_a(s)$ ,  $K = 1$
- 2)  $H_h(s)$ ,  $\omega_p = 1.15148675$ ,  $Q_p = 0.61625492$
- 3)  $H_h(s)$ ,  $\omega_p = 0.95360261$ ,  $Q_p = 0.51157670$
- 4)  $H_h(s)$ ,  $\omega_p = 0.98559505$ ,  $Q_p = 0.53174411$
- 5)  $H_h(s)$ ,  $\omega_p = 1.02900047$ ,  $Q_p = 0.56136078$
- 6)  $H_h(s)$ ,  $\omega_p = 1.07702300$ ,  $Q_p = 0.60099034$
- 7)  $H_h(s)$ ,  $\omega_p = 1.11337090$ ,  $Q_p = 0.65696211$
- 8)  $H_h(s)$ ,  $\omega_p = 1.07821613$ ,  $Q_p = 0.79159298$
- 9)  $H_m(s)$ ,  $\omega_z = 3.1335590$
- 10)  $H_m(s)$ ,  $\omega_z = 3.7630714$
- 11)  $H_m(s)$ ,  $\omega_z = 4.7859023$
- 12)  $H_m(s)$ ,  $\omega_z = 6.0011218$
- 13)  $H_m(s)$ ,  $\omega_z = 7.3087231$
- 14)  $H_m(s)$ ,  $\omega_z = 8.6935655$

where each elementary function type is followed by its parameters.

Diagrams containing piecewise linear Bode plots accompanied by the exact plots are given in Figs. 12 and 13 for the amplitude and the phase response, respectively. Value

Fig. 12.  $H_{vbl}(s)$ , amplitude response.Fig. 13.  $H_{vbl}(s)$ , phase response.

$\omega_n$  in the diagrams indicates the normalization frequency. In the amplitude response, the diagram is polished manually, such that the arrows that would indicate the infinite response are replaced by the lines that spread to the diagram border. Regardless the fact that the poles and zeros are close, the approximation is acceptable. Numerous undershoots caused by low Q factor values of the pole are observed around  $\omega \approx \omega_n$ . Infinite response caused by zero pairs on the imaginary axes are correctly handled. In phase response, correct handling of discontinuities caused by the imaginary axis zero pairs could be observed.

## VI. FUTURE WORK

At present stage, the described algorithm is implemented in a command line interface based program. The use of such program requires some sort of computer usage competence, which is not common to the majority of users nowadays. Thus, to increase availability, a graphical user interface is planned to be created. Another direction of program development is to provide automatic creation of the canonical form of (1) on the basis of the transfer function specified as a rational

function of  $s$ . Finally, the algorithm is intended to be included in a comprehensive symbolic analysis software suite for linear systems.

## VII. CONCLUSION

In this paper, an algorithm for creating piecewise linear Bode plots of amplitude and phase frequency responses of lumped parameter linear systems is proposed. The algorithm is complete in the sense that it covers for poles and zeros placed at arbitrary locations in the entire complex plane.

The transfer function is factored into a specified canonical form, as products of elementary transfer functions. Thirteen elementary transfer functions are defined, and plotting piecewise linear amplitude and phase Bode plots is defined for each of them. It is shown that the thirteen elementary transfer functions have their roots in only five generic transfer function types. Piecewise linear Bode plots are depicted for all of these five generic types, and specified for other elementary transfer functions by appropriate mirroring.

The algorithm to generate piecewise linear Bode plots is based upon introducing nodes in both the amplitude and the phase frequency response. The nodes are defined as points where the response, either amplitude or phase, changes its slope. For each of the elementary functions the list of nodes it introduces both in the amplitude and in the phase response is presented, as well as the algorithm how to modify the node altitudes according to the effects caused by the elementary transfer function.

Finally, the algorithm to generate the Bode plots is described as a three stage algorithm. In the first stage, just after the input file parsing, frequencies of the nodes specified by the elementary functions of the transfer function factorization into canonical form are collected, and the border frequencies for the plots are included in the list of node frequencies. In the second stage, node altitudes are updated according to specifications given for each of the elementary transfer function types. In the third stage, overshoots and undershoots in the amplitude response created by complex conjugate pairs of pole or zeros are resolved, separately for the pairs outside the imaginary axis, and for the pairs located on the imaginary axis, creating linear segment patches for the overshoots or undershoots or arrow patches that indicate infinite logarithmic amplitude response, respectively. Output of the program is a diagram and a data structure that allows the user to create own diagrams to tweak the appearance if required.

An issue worth addressing is computational complexity of the proposed algorithm in comparison with standard general purpose tools that might provide Bode plots. The first difference to be underlined is in the size of the data set that describes the plot: in the case of piecewise linear Bode plot the size is related to the number of poles and zeros, and it is much smaller than the data sets that characterize standard smooth frequency response plots. Actually, this is the main reason that Bode plots are still in use today: the reduced data set is much easier to handle by human cognitive capacities in the design process. In smooth frequency response plots, the

number of poles and zeros of the transfer function is not related to the data set size, instead it is determined by the number of data points required to provide the diagram detailed enough, frequently containing excessive number of points covering parts of the frequency response where not much happens. Assuming that computational complexity to generate one data point is about the same in both approaches, the conclusion is that the asymptotic Bode plot approach requires significantly reduced computational effort. Furthermore, smooth phase response plots require numerous arctangent (actually,  $\text{atan2}$ ) computations, which are computationally more demanding than the arithmetic required by the asymptotic plots. However, it is worth to note that asymptotic Bode plots require transfer function factorization, which standard approach does not. However, factorization is required in the design process anyhow, to properly understand dynamics and to adequately place poles and zeros of the compensating regulator. Finally, with modern computers computational efficiency in creating the frequency response plot should not be a critical issue nowadays, and the main reason for using the asymptotic Bode plot approach is in its cognitive value, clear and concise presentation of the system dynamics.

#### REFERENCES

- [1] L. W. Nagel, *SPICE2: A computer program to simulate semiconductor circuits*, ERL Memo ERL-M520, 1975.
- [2] A. Vladimirescu, *The SPICE book*. John Wiley & Sons, Inc., 1994.
- [3] Wikipedia contributors, "Bode plot," [Online]. Available: [https://en.wikipedia.org/wiki/Bode\\_plot](https://en.wikipedia.org/wiki/Bode_plot)
- [4] G. F. Franklin, J. D. Powell, A. Emami-Naeini, *Feedback control of dynamic systems*, 3rd ed., Addison-Wesley, 1994
- [5] K. Ogata, *Modern control engineering*, 5th ed., Pearson Education, 2010
- [6] GNU Octave, [Online]. Available: <https://www.gnu.org/software/octave/>
- [7] R. D. Middlebrook, "Low Entropy Expressions: The Key to Design-Oriented Analysis," IEEE Frontiers in Education Conference, 1991 Proceedings, pp. 399–403, Sept. 1991.
- [8] R. D. Middlebrook, "Methods of Design-Oriented Analysis: The Quadratic Equation Revisited," IEEE Frontiers in Education Conference, 1992 Proceedings, pp. 95–102, Nov. 1991.
- [9] R. W. Erickson, *Bode Diagrams of Transfer Functions and Impedances*, ECEN 2260 Supplementary Notes, [Online]. Available: <http://eceee.colorado.edu/ecen2270/materials/Bodenotes.pdf>
- [10] R. W. Erickson and D. Maksimović, *Fundamentals of power electronics*, 2nd ed., Kluwer Academic Publishers, 2001
- [11] V. Vorperian, *Fast analytical techniques for electrical and electronic circuits*, Cambridge University Press, 2002.
- [12] C. K. Alexander and M. N. O. Sadiku, *Fundamentals of electric circuits*, 2nd ed., McGraw-Hill, 2004.
- [13] A. Agarwal, J. H. Lang *Foundations of analog and digital electronic circuits*, Elsevier, 2005.
- [14] Python Programming Language — Official Website, [Online]. Available: <http://www.python.org/>.
- [15] NumPy [Online]. Available: <http://www.numpy.org/>
- [16] SciPy [Online]. Available: <https://www.scipy.org/>
- [17] matplotlib, [Online]. Available: <http://matplotlib.org/>
- [18] Maxima, a Computer Algebra System, [Online]. Available: <http://maxima.sourceforge.net/>
- [19] SymPy, [Online]. Available: <http://www.sympy.org/en/index.html>
- [20] V. B. Litovski, "Synthesis of Monotonic Passband Sharp Cutoff Filters with Constant Group Delay Response," *IEEE Transactions on Circuits and Systems*, vol. CAS-26, no. 8, pp. 597–602, Aug. 1979.
- [21] V. B. Litovski, D. P. Milovanović, "New solution for the ideal filter approximation problem," *IEE Proceedings G (Electronic Circuits and Systems)*, vol. 130, no. 4, pp. 161–163, Aug. 1983.